BUCKLING THEORY IN SOLID STRUCTURE WITH SMALL THICKNESS (Part I)
—General Formulation of Problem—

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A consistent methodology for the analysis of buckling phenomena in three dimensional solids is presented in this Part I and a straightforward application of this theory to a simple structure will be treated in the forthcoming Part II. The main objective of the work in this Part I is to derive the asymptotic equation from the complicate three dimensional stability equation so that we may obtain the buckling solution in the strained solids analytically and/or numerically very easily. The desired accurate equations are obtained through the regular asymptotic technique.

Key Words: Buckling, Stability, Bifurcation, Lagrangian Description, Physical Component, Function Space

1. INTRODUCTION

The buckling phenomenon usually occurs in such solid structures as bar, column, plate and shell etc. when those solid structures are strained beyond the critical load. Those bar, column, plate and shell are basic components of all kinds of machinery like automobile, aircraft, ship etc. as well as of the architectural structures like bridge, house etc.. It is not wanted for this buckling to happen during the actual loading process, but such a phenomenon is very interesting to the observer. The term "buckling" here is a generic one and incorporates all abrupt changes in the deformation pattern of a structure, occuring in the course of a loading process. In mathematical terms the corresponding phenomenon is called "bifurcation" and it involves the loss of uniqueness in the solution of the nonlinear governing equations for the pertaining boundary value problem describing the deformation of the structure in question.

Although the first buckling study goes back to Euler, i.e., the so-called Euler's beam theory, the theoretical mathematical foundation for the theory of structural stability as a bifurcation problem is due to the recent works of Koiter (1945) and Hill (1957, 1958). Thereafter, vast efforts were made to solve the buckling problem. However, the results were not so much satisfactory (see Timosenko (1961), Hutchinson (1974)). The main difficulty in solving the bifurcation equation is due to the nonlinearity (largely, the geometric nonlinearity and the material nonlinearity). Hence, it is extremely difficult to get the mathematical closed form solution of a specific buckling problem even though it has a simple geometry. The numerical technique (e.g., finite element method etc.) has been helpful in the analysis.

There are three steps in the buckling analysis. The first step is to analyse the prebuckling (or primary) state. The next step is to determine the critical marginal state including the critical load when the solid structural state starts to become unstable. The final step is to analyse the postbuckling behavior of the solid structure, but this postbuckling analysis is not necessarily required for the practical use even though it is very interesting phenomenally. The most important result in the buckling analysis is the critical condition which can be used for the structural design. Hence, most efforts in the analysis are concentrated on the second step.

The main purpose of this work is to derive the unique general consistent method to determine the critical condition which is independent of the diversified classical nonlinear theories and can be used for the numerical solution as well as for the analytical solution. The similar effort was tried by Triantafyllidis and Kwon (1987). And its practical application was done by Donoghue, Stevenson, Kwon and Triantafyllidis (1989) in the puckering analysis.

In place of the classical approach, in which a two dimensional nonlinear theory (derived from the three dimensional governing equations of the solid) is linearized about the critical load, the present method starts from the bifurcation equation of the three dimensional solid (which have been obtained by linearization about the critical load of the same three dimensional governing equations for the nonlinear solid in question) and subsequently takes the limits as the structure thickness h tends to zero, following an asymptotic technique.

2. GENERAL FEATURES OF BUCKLING EQUATION

The starting point for the analysis of solid mechanics is always the following three dimensional equation governing the equilibrium state of solid, adopting the full Lagrangian description

$$(\tau^{ij}+\tau^{kj}u^{i}{}_{,k})_{,j}=0 \text{ in } V \tag{1}$$

with the following boundary conditions

$$T^{i}=(\tau^{ij}+\tau^{kj}u^{i}{}_{,k})N_{j} \text{ (for the specified traction) or}$$
$$\delta u_{i}=0 \text{ (for the specified displacement) on } \partial V \tag{2}$$

Equivalently, if the deformed current configuration coincides with the reference one, (in other words, if we take the current

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deformed configuration as a reference one), the above equation takes

$$\sigma^{ij}{}_{,j}=0 \text{ or}$$
$$\partial\sigma^{ij}/\partial\theta^j+\Gamma^j_{mj}\sigma^{mi}+\Gamma^i_{mj}\sigma^{jm}=0 \text{ in } V \quad (3)$$

with $t^i=\sigma^{ij}n_j$(for specified traction) or $\delta u_i=0$(for specified displacement) on ∂V, where ',' means the covariant derivative. We need another expression corresponding to the above equation for the numerical analysis (especially, finite element method). In such a case, we can set up the following principle of virtual work.

$$\int_V (\boldsymbol{S} : \delta\boldsymbol{E})\,dV=\int_{\partial V}\boldsymbol{T}\cdot\delta\boldsymbol{u}ds \quad (4)$$

In all the cases in the above equations, we define the following notation coventions.

$\boldsymbol{S}=J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{F}^{-1})^T$: Second Piola Kirchhoff Stress
$\boldsymbol{\tau}=J\boldsymbol{\sigma}$: Kirchhoff Stress($\tau^{ij}=S^{ij}$)
$\boldsymbol{\sigma}=\sigma^{ij}\boldsymbol{G}^i\boldsymbol{G}^j$: Current Cauchy Stress (5)
$\boldsymbol{E}=(1/2)\{\nabla\boldsymbol{u}+\boldsymbol{u}\nabla+(\nabla\boldsymbol{u})\cdot(\boldsymbol{u}\nabla)\}$: Lagrangian Strain Tensor
$\boldsymbol{u}=u_k\boldsymbol{G}^k=u^k\boldsymbol{G}_k$: displacement vector
$\boldsymbol{G}_i=\partial\boldsymbol{P}/\partial\theta^i$: reference covariant vector
$\boldsymbol{T}=T^i\boldsymbol{G}_i$: Pseudo traction vector per unit reference area
Γ^i_{mj} : Christoffel Symbols
J : Jacobian
and δu_i are admissible variations satisfying the boundary conditions.

Sometimes, we need a little variation of the above equation for the special analysis : for example, the analysis of the large elastic or plastic deformation. In such a case, usually we have the incremental constitutive equation. Hence, with rates of change denoted by a dot, i.e., the dot denotes the derivative with respect to some monotonically increasing time-like parameter, which is also referred to as an increment of the quantity in question, then we have the incremental form of the governing equation as

$$(\dot\tau^{ij}+\dot\tau^{kj}u^i_{,k}+\tau^{ki}\dot u^i_{,k})_{,j}=0 \text{ in } V \quad (6)$$

with $\dot T^i=(\dot\tau^{ij}+\dot\tau^{kj}u^i_{,k}+\tau^{kj}\dot u^i_{,k})N_j$ or $\delta\dot u_i=0$ on ∂V

Equivalently, we may have the following incremental form of principle of virtual work,

$$\int_V(\dot\tau^{ij}\delta E_{ij}+\tau^{ij}\dot u^k_{,i}\delta u_{k,j})\,dV=\int_{\partial V}\dot T^i\delta u_i dS \quad (7)$$

More specifically, for the buckling (bifurcation) analysis of elastic and plastic deformations of solids, we should have the corresponding bifurcation equation. Now, at any state of deformation characterized by $\boldsymbol{u}$, suppose that the bifurcation is possible, so that for a given load increment $\dot\lambda$ (λ : load parameter), there are at least two solutions $\dot{\boldsymbol{u}}^a$ and $\dot{\boldsymbol{u}}^b$, one of which is the prebuckling solution (fundamental or principle solution). Introducing the following notation convention for the difference between two possible solutions a and b in any field quantity,

$$\Delta(\cdot)=(\cdot)^b-(\cdot)^a$$

i.e., $\Delta\dot{\boldsymbol{u}}=\dot{\boldsymbol{u}}^b-\dot{\boldsymbol{u}}^a$
$\Delta\dot{\boldsymbol{\tau}}=\dot{\boldsymbol{\tau}}^b-\dot{\boldsymbol{\tau}}^a$ etc.

the incremental stability (buckling) equation linearized around the critical state is derived as

$$(\Delta\dot\tau^{ij}+u^i_{,k}\Delta\dot\tau^{kj}+\tau^{kj}\Delta\dot u^i_{,k})_{,j}=0 \text{ in } V \quad (8)$$

with $\Delta\dot T^i=(\Delta\dot\tau^{ij}+u^i_{,k}\Delta\dot\tau^{kj}+\tau^{kj}\Delta\dot u^i_{,k})N_j$ or $\delta\Delta\dot u_i=0$ on ∂V

And the corresponding principle of virtual work is

$$0=\int_{\partial V}\Delta\dot T^i\Delta\dot u_i dS=\int_V(\Delta\dot\tau^{ij}\Delta\dot E_{ij}+\tau^{ij}\Delta\dot u^k{}_{,i}\Delta\dot u_{k,j})\,dV \quad (9)$$

Attention is next focused on the constitutive equations. Of interest in all stability analyses for rate independent solids (hyperelastic, hypoelastic or more generally elastoplastic solids) is the rate form of their constitutive equation, which most often is given in the form of the fourth order incremental moduli tensor L^{ijkl} relating the convective rate of the Kirchhoff stress $\dot\tau^{ij}$ to the strain rate tensor D_{kl} in the current configuration (or equivalently relating the rate of the Second Piola Kirchhoff Stress $\dot S^{ij}$ to the rate of the Lagrange-Green Strain $\dot E_{kl}$ in the reference configuration), namely

$$\dot\tau^{ij}=L^{ijkl}D_{kl} \text{ (or } \dot S^{ij}=L^{ijkl}\dot E_{kl}) \quad (10)$$

For most rate independent materials encountered in the applications, the incremental moduli tensor has the following symmetries

$$L^{ijkl}=L^{klij}=L^{jikl}=L^{ijlk} \quad (11)$$

It should be noted at this point that the first equation in the above equaiton is very important, for it insures the existence of a bifurcation functional, say F, quadratic in terms of the bifurcation eigenmode v_i, which loses its positive definiteness for the first time, as the load parameter increases away from zero, at the onset of the first bifurcation instability. Following Hill(1956, 1958), Hutchinson(1974), the bifurcation functional F is given by

$$F(\lambda,\boldsymbol{v})=(1/2)\int_V \mathrm{L}^{ijkl}\Delta\dot u_{i,j}\Delta\dot u_{k,l}dV \ ; \ \mathrm{L}^{ijkl}$$
$$=L^{sjql}F_{rs}F_{pq}G^{ri}G^{pk}+\tau^{lj}G^{ki} \quad (12)$$

where the covariant differentiation is with respect to the reference metric. At the onset of the first variation of F, $\delta F=0$. The above functional form of F is valid for the case of dead surface and body loads. For the case of configuration dependent loading, some additional surface terms will be required for F. The minimum eigenvalue of F, say B, is defined as

$$\mathrm{B}(\lambda)=\min[2F(\lambda,\boldsymbol{v})/\|\boldsymbol{v}\|^2] \ ; \ \|\boldsymbol{v}\|^2=\int_V G^{ij}\Delta\dot u_i\Delta\dot u_j dV \quad (13)$$

and for a given solid geometry $\mathrm{B}(\lambda)>0$ for $\lambda<\lambda_c$, while $\mathrm{B}(\lambda)=0$ for $\lambda=\lambda_c$ the lowest critical load for F. The eigenvalue B and the corresponding eigenfunction $\boldsymbol{v}$ are found from the variational equation

$$\delta F=\mathrm{B}\delta(\|\Delta\dot{\boldsymbol{u}}\|^2/2) \quad (14)$$

So, for B=0, the corresponding stability equation is

$$\delta F=0 \quad (15)$$

3. ASYMPTOTIC FEATURES OF BUCKLING EQUATION

In this section our attention is focused on the specified case, i.e., the bifurcation phenomena of the solid structures of bars, columns, plates and shells with traction free surfaces. These solid structures commonly have a small geometric characteristic parameter of a thickness (namely, h). And we will derive the asymptotic features of the stability equation with respect to this small parameter. We will always consider the reference configuration with a uniform thickness h. This reference configuration will be either the deformed current configuration or the stress free initial configuration.

The derivation will be developed in a pointwise manner. The bifurcation equation corresponding to the bifurcation functional at the critical state (marginal state) where the minimum eigenvalue B is zero will be

$$(\mathrm{L}^{ijkl}\Delta\dot u_{l,k})_{,i}=0 \quad \text{in } V$$

& $\quad \mathrm{L}^{ijkl}\Delta\dot{u}_{l,k}n_i=0 \qquad \text{on } \partial V \qquad (16)$

where the comma ',' means the covariant derivative i.e.,

$$\Delta\dot{u}_{l,k}=\partial\Delta\dot{u}_l/\partial\theta^k-\Delta\dot{u}_m\Gamma^m_{lk}$$
$$\Delta\dot{T}^{\alpha j}{}_{,\alpha}=\partial\Delta\dot{T}^{\alpha j}/\partial\theta^\alpha+\Gamma^\alpha_{m\alpha}\Delta\dot{T}^{mj}+\Gamma^j_{m\alpha}\Delta\dot{T}^{\alpha m} \qquad (17)$$

Now, we use the coordinate system (θ^α,ζ) for the Lagrangian description, where $\theta^\alpha=(\theta^1,\theta^2)$ describes the reference midgeometry which is given for a specific configuration and ζ is the thickness coordinate along the normal to the reference midgeometry of the reference solid configuration.

In order to obtain a proper equation for the asymptotic analysis, we nondimensionalize the coordinates and normalize the thickness coordinate as

$$\theta^\alpha=\theta^\alpha/\mathrm{L},\ \zeta^*=\zeta-g(\theta^\alpha),\ \ \zeta^*=\zeta^*/\mathrm{L}\,(-h/2\le\zeta^*\le h/2)$$
& $\quad \xi=\zeta^*/\varepsilon$, or $\zeta^*=\varepsilon\xi \ \ (-1/2\le\xi\le1/2) \qquad (18)$

where $\varepsilon=h/\mathrm{L}$ (L ; a characteristic length of the reference configuration) and $g(\theta^\alpha)$ is a coordinate of the reference midgeometry.

Using the above parametrization, the bifurcation Eq. (16) will be

$$\varepsilon^2\{(\mathrm{L}^{\alpha j\beta l}\Delta\dot{u}_{l,\beta})_{,\alpha}-(\mathrm{L}^{\alpha j3l}\Delta\dot{u}_m\Gamma^m_{l3})_{,\alpha}+\Gamma^j_{m3}(\mathrm{L}^{3m\beta l}\Delta\dot{u}_{l,\beta}-\mathrm{L}^{3m3l}\Delta\dot{u}_n\Gamma^n_{l3})\}+\varepsilon\{(\mathrm{L}^{\alpha j3l}\partial\Delta\dot{u}_l/\partial\xi)_{,\alpha}+(\partial/\partial\xi)(\mathrm{L}^{3j\beta l}\Delta\dot{u}_{l,\beta}-\mathrm{L}^{3j3l}\Delta\dot{u}_m\Gamma^m_{l3})+\Gamma^j_{m3}\mathrm{L}^{3m3l}\partial\Delta\dot{u}_l/\partial\xi\}(\partial/\partial\xi)(\mathrm{L}^{3j3l}\partial\Delta\dot{u}_l/\partial\xi)+(\partial/\partial\xi)=0$$

(with $\Gamma^i_{jk}\equiv\Gamma^j_{jk}\mathrm{L}$) for $-1/2\le\xi\le1/2$

& $\quad n_i\{\varepsilon(\mathrm{L}^{ij\beta l}\Delta\dot{u}_{l,\beta}-\mathrm{L}^{ij3l}\Delta\dot{u}_m\Gamma^m_{l3})+\mathrm{L}^{ij3l}\partial\Delta\dot{u}_l/\partial\Delta\dot{u}_l/\partial\xi\}=0$ at $\xi=\pm1/2 \qquad (19)$

Now, under the assumption that there is no drastic change in the prebuckling state and the buckling mode, a straightforward regular perturbation technique will be adopted for the analysis. For the further asymptotic analysis, we should have a proper prebuckling expansions with respect to the thickness parameter ε.

i.e., $\mathrm{L}^{ijkl}=\overset{0}{\mathrm{L}}{}^{ijkl}+\overset{1}{\mathrm{L}}{}^{ijkl}\varepsilon++\overset{2}{\mathrm{L}}{}^{ijkl}\varepsilon^2+\overset{3}{\mathrm{L}}{}^{ijkl}\varepsilon^3+\cdots$

$$\Gamma^i_{jk}=\overset{0}{\Gamma}{}^i_{jk}+\overset{1}{\Gamma}{}^i_{jk}\varepsilon+\overset{2}{\Gamma}{}^i_{jk}\varepsilon^2+\overset{3}{\Gamma}{}^i_{jk}\varepsilon^3+\cdots \qquad (20)$$

& noting $n_i=\delta_{i3}$

And also we assume the buckling modes as

$$\Delta\dot{u}_l=\overset{0}{u}_l+\overset{1}{u}_l\varepsilon+\overset{2}{u}_l\varepsilon^2+\overset{3}{u}_l\varepsilon^3+\cdots$$
$$\lambda=\overset{0}{\lambda}+\overset{1}{\lambda}\varepsilon+\overset{2}{\lambda}\varepsilon^2+\overset{3}{\lambda}\varepsilon^3+\cdots \qquad (21)$$

$$\Delta\dot{u}_{l,\beta}=\partial\Delta\dot{u}_l/\partial\theta_\beta-\Delta\dot{u}_m\Gamma^m{}_{l\beta}=\overset{0}{u}_{l,\beta}+\overset{1}{u}_{l,\beta}\varepsilon+\overset{2}{u}_{l,\beta}\varepsilon^2+\overset{3}{u}_{l,\beta}\varepsilon^3+\cdots$$

with definitions

$$\overset{0}{u}_{l,\beta}=\partial\overset{0}{u}_l/\partial\theta^\beta-\overset{0}{u}_m\overset{0}{\Gamma}{}^m_{l\beta}$$
$$\overset{1}{u}_{l,\beta}=\partial\overset{1}{u}_l/\partial\theta^\beta-(\overset{0}{u}_m\overset{1}{\Gamma}{}^m_{l\beta}+\overset{1}{u}_m\overset{0}{\Gamma}{}^m_{l\beta})$$
$$\overset{2}{u}_{l,\beta}=\partial\overset{2}{u}_l/\partial\theta^\beta-(\overset{0}{u}_m\overset{2}{\Gamma}{}^m_{l\beta}+\overset{1}{u}_m\overset{1}{\Gamma}{}^m{}_{l\beta}+\overset{2}{u}_m\overset{0}{\Gamma}{}^m_{l\beta})$$
$$\overset{3}{u}_{l,\beta}=\partial\overset{3}{u}_l/\partial\theta^\beta-(\overset{0}{u}_m\overset{3}{\Gamma}{}^m_{l\beta}+\overset{1}{u}_m\overset{2}{\Gamma}{}^m_{l\beta}+\overset{2}{u}_m\overset{1}{\Gamma}{}^m_{l\beta}+\overset{3}{u}_m\overset{0}{\Gamma}{}^m_{l\beta})$$

etc.

and

$$\Delta\dot{T}^{\alpha j}{}_{,\alpha}=\partial\Delta\dot{T}^{\alpha j}/\partial\theta^\alpha+\Gamma^\alpha_{m\alpha}\Delta\dot{T}^{mj}+\Gamma^j_{m\alpha}\Delta\dot{T}^{\alpha m}$$
$$=\overset{0}{T}{}^{\alpha j}{}_{,\alpha}^{0}+\overset{1}{T}{}^{\alpha j}{}_{,\alpha}^{1}\varepsilon+\overset{2}{T}{}^{\alpha j}{}_{,\alpha}^{2}\varepsilon^2+\overset{3}{T}{}^{\alpha j}{}_{,\alpha}^{3}\varepsilon^3+\cdots \qquad (22)$$

with

$$\overset{0}{T}{}^{\alpha j}{}_{,\alpha}^{0}=\partial\overset{0}{T}{}^{\alpha j}/\partial\theta^\alpha+\overset{0}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{T}{}^{mj}+\overset{0}{\Gamma}{}^j_{m\alpha}\overset{0}{T}{}^{\alpha m}$$
$$\overset{1}{T}{}^{\alpha j}{}_{,\alpha}^{1}=\overset{1}{T}{}^{\alpha j}{}_{,\alpha}^{0}+\overset{1}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{T}{}^{mj}+\overset{1}{\Gamma}{}^j_{m\alpha}\overset{0}{T}{}^{\alpha m}$$
$$\overset{2}{T}{}^{\alpha j}{}_{,\alpha}^{2}=\overset{2}{T}{}^{\alpha j}{}_{,\alpha}^{0}+\overset{1}{\Gamma}{}^\alpha{}_{m\alpha}\overset{1}{T}{}^{mj}+\overset{1}{\Gamma}{}^j_{m\alpha}\overset{1}{T}{}^{\alpha m}+\overset{2}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{T}{}^{mj}+\overset{2}{\Gamma}{}^j_{m\alpha}\overset{0}{T}{}^{\alpha m}$$
$$\overset{3}{T}{}^{\alpha j}{}_{,\alpha}^{3}=\overset{3}{T}{}^{\alpha j}{}_{,\alpha}^{0}+\overset{1}{\Gamma}{}^\alpha_{m\alpha}\overset{2}{T}{}^{mj}+\overset{1}{\Gamma}{}^j_{m\alpha}\overset{2}{T}{}^{\alpha m}+\overset{2}{\Gamma}{}^\alpha_{m\alpha}\overset{1}{T}{}^{mj}+\overset{2}{\Gamma}{}^j_{m\alpha}\overset{1}{T}{}^{\alpha m}+\overset{3}{\Gamma}{}^j_{m\alpha}\overset{0}{T}{}^{mj}+\overset{3}{\Gamma}{}^j_{m\alpha}\overset{0}{T}{}^{\alpha m}$$

etc.

And then, the straightforward expansion of the bifurcation equation will give us the following asymptotic solutions.

The lowest order solution is,

$$(\overset{0}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\alpha}^{0}=0 \qquad (23)$$

where $\overset{0}{G}{}^{\alpha j\beta l}=\overset{0}{\mathrm{L}}{}^{\alpha j\beta l}-\overset{0}{\mathrm{L}}{}^{\alpha j3m}(\overset{0}{\mathrm{L}}{}^{3n3m})^{-1}\overset{0}{\mathrm{L}}{}^{3n\beta l}$

and the solution is

$$\partial\overset{0}{u}_l/\partial\xi=0 \quad \text{or} \quad \overset{0}{u}_l=\overset{0}{w}_l(\theta^\alpha)$$
$$\partial\overset{1}{u}_l/\partial\xi=-(\overset{0}{\mathrm{L}}{}^{3j3l})^{-1}(\overset{0}{\mathrm{L}}{}^{3j\beta k}\overset{0}{u}_{k,\beta}-\overset{0}{\mathrm{L}}{}^{3j3k}\overset{0}{u}_n\overset{0}{\Gamma}{}^n_{k3})=\overset{1}{\mathrm{B}}_l(\theta^\alpha)$$

The first order solution is,

$$\int_{-1/2}^{1/2}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\alpha}^{0}+\overset{1}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta}+\overset{1}{\Gamma}{}^j_{m\alpha}\overset{0}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta}\}d\xi=0$$

where,

$$\overset{1}{G}{}^{\alpha j\beta l}=\overset{1}{\mathrm{L}}{}^{\alpha j\beta k}-\overset{0}{\mathrm{L}}{}^{\alpha j3l}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{\mathrm{L}}{}^{3m\beta k}-\overset{1}{\mathrm{L}}{}^{\alpha j3l}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{0}{\mathrm{L}}{}^{3m\beta l}+\overset{0}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{\mathrm{L}}{}^{3m3p}(\overset{0}{\mathrm{L}}{}^{3n3p})^{-1}\overset{0}{\mathrm{L}}{}^{3n\beta l} \qquad (24)$$

and the solution is

$$\partial\overset{2}{u}_l/\partial\xi=-(\overset{0}{\mathrm{L}}{}^{3j3l})^{-1}[\{\overset{1}{\mathrm{L}}{}^{3j\beta k}-\overset{1}{\mathrm{L}}{}^{3j3m}(\overset{0}{\mathrm{L}}{}^{3n3m})^{-1}\overset{0}{\mathrm{L}}{}^{3n\beta k}\}u^0_{k,\beta}+\overset{0}{\mathrm{L}}{}^{3j\beta k}\overset{1}{u}_{k,\beta}-\overset{0}{\mathrm{L}}{}^{3j3k}\overset{0}{u}_m\overset{1}{\mathrm{L}}{}^m_{k3}-\overset{0}{\mathrm{L}}{}^{3j3k}\overset{1}{u}_m\overset{0}{\mathrm{L}}{}^m_{k3}]$$

The second order solution is,

$$\int_{-1/2}^{1/2}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{2}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{2}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\alpha}^{0}+\overset{1}{\Gamma}{}^\alpha_{m\alpha}(\overset{0}{G}{}^{mj\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta})+\overset{1}{\Gamma}{}^j_{m\alpha}(\overset{0}{G}{}^{\alpha m\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta})+\overset{2}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta}+\overset{2}{\Gamma}{}^j_{m\alpha}\overset{0}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta}\}d\xi$$
$$=\int_{-1/2}^{1/2}[\{\overset{0}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{a}{}^m\}_{,\alpha}^{0}+\overset{0}{\Gamma}{}^j_{m3}\overset{1}{a}{}^m]\,d\xi \qquad (25)$$

where

$$\overset{2}{G}{}^{\alpha j\beta l}=\overset{2}{\mathrm{L}}{}^{\alpha j\beta k}-\overset{0}{\mathrm{L}}{}^{\alpha j3l}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{2}{\mathrm{L}}{}^{3m\beta l}-\overset{2}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{0}{\mathrm{L}}{}^{3m\beta l}-\overset{1}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{\mathrm{L}}{}^{3m\beta k}+\overset{0}{\mathrm{L}}{}^{\alpha j3l}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{\mathrm{L}}{}^{3m3k}(\overset{0}{\mathrm{L}}{}^{3p3m})^{-1}\overset{0}{\mathrm{L}}{}^{3p\beta l}+\overset{1}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{1}{\mathrm{L}}{}^{3m3n}(\overset{0}{\mathrm{L}}{}^{3p3n})^{-1}\overset{0}{\mathrm{L}}{}^{3p\beta l}+\overset{0}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{2}{\mathrm{L}}{}^{3m3p}(\overset{0}{\mathrm{L}}{}^{3q3p})^{-1}-\overset{0}{\mathrm{L}}{}^{3q\beta l}\overset{0}{\mathrm{L}}{}^{\alpha j3k}(\overset{0}{\mathrm{L}}{}^{3m3k})^{-1}\overset{0}{\mathrm{L}}{}^{3m3n}(\overset{0}{\mathrm{L}}{}^{3p3n})^{-1}\overset{1}{\mathrm{L}}{}^{3p3q}(\overset{0}{\mathrm{L}}{}^{3r3q})^{-1}\overset{0}{\mathrm{L}}{}^{3r\beta l}$$

and

$$\overset{1}{a}{}^m=\int_{-1/2}^{\xi}\{(\overset{0}{G}{}^{\gamma m\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\gamma m\beta l}\overset{0}{u}_{l,\beta})_{,\gamma}^{0}+\overset{1}{\Gamma}{}^r_{nr}\overset{0}{G}{}^{nm\beta l}\overset{0}{u}_{l,\beta}+\overset{1}{\Gamma}{}^m_{nr}\overset{0}{G}{}^{rn\beta l}\overset{0}{u}_{l,\beta}\}d\xi$$

and the solution is

$$\partial\overset{3}{u}_l/\partial\xi=-(\overset{0}{\mathrm{L}}{}^{3m3l})^{-1}[\overset{1}{a}{}^m+\overset{0}{\mathrm{L}}{}^{3m\beta k}\overset{2}{u}_{k,\beta}+\{\overset{1}{\mathrm{L}}{}^{3m\beta k}-\overset{1}{\mathrm{L}}{}^{3m3p}(\overset{0}{\mathrm{L}}{}^{3r3p})^{-1}\overset{0}{\mathrm{L}}{}^{3r\beta k}\}\overset{1}{u}_{k,\beta}+\{\overset{2}{\mathrm{L}}{}^{3m\beta k}-\overset{1}{\mathrm{L}}{}^{3m3p}(\overset{0}{\mathrm{L}}{}^{3r3p})^{-1}(\overset{1}{\mathrm{L}}{}^{3r\beta k}-\overset{1}{\mathrm{L}}{}^{3r3n}(\overset{0}{\mathrm{L}}{}^{3s3n})^{-1}\overset{0}{\mathrm{L}}{}^{3s\beta k})-(\overset{2}{\mathrm{L}}{}^{3m3p}(\overset{0}{\mathrm{L}}{}^{3r3p})^{-1}\overset{0}{\mathrm{L}}{}^{3r\beta k}\}\overset{0}{u}_{k,\beta}-\overset{0}{\mathrm{L}}{}^{3m3k}(\overset{0}{u}_n\overset{2}{\Gamma}{}^n_{k3}+\overset{1}{u}_n\overset{1}{\Gamma}{}^n_{k3}+\overset{2}{u}_n\overset{0}{\Gamma}{}^n_{k3})]$$

The third order solution is,

$$\int_{-1/2}^{1/2}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{3}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{2}{u}_{l,\beta}+\overset{2}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{3}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\alpha}^{0}$$
$$+\overset{1}{\Gamma}{}^\alpha_{m\alpha}(\overset{0}{G}{}^{mj\beta l}\overset{2}{u}_{l,\beta}+\overset{1}{G}{}^{mj\beta l}\overset{1}{u}_{l,\beta}+\overset{2}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta})$$
$$+\overset{1}{\Gamma}{}^j_{m\alpha}(\overset{0}{G}{}^{\alpha m\beta l}\overset{2}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha m\beta l}\overset{1}{u}_{l,\beta}+\overset{2}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta})$$
$$+\overset{2}{\Gamma}{}^\alpha_{m\alpha}(\overset{0}{G}{}^{mj\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta})$$
$$+\overset{2}{\Gamma}{}^j_{m\alpha}(\overset{0}{G}{}^{\alpha m\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta})+\overset{3}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta}$$
$$+\overset{3}{\Gamma}{}^j_{m\alpha}\overset{0}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta}\}d\xi=\int_{-1/2}^{1/2}[\{\overset{0}{\mathrm{L}}{}^{\alpha j3k}-(\overset{0}{\mathrm{L}}{}^{3m3k})^{-2}\overset{2}{a}{}^m\}^0_{,\alpha}$$
$$+\overset{1}{\Gamma}{}^\alpha_{m\alpha}\overset{0}{\mathrm{L}}{}^{3m3k}(\overset{0}{\mathrm{L}}{}^{3p3k})^{-1}\overset{1}{a}{}^p\overset{1}{\mathrm{L}}{}^j_{m\alpha}\overset{0}{\mathrm{L}}{}^{\alpha m3k}(\overset{0}{\mathrm{L}}{}^{3p3k})^{-1}\overset{1}{a}{}^p$$
$$+\overset{0}{\Gamma}{}^j_{m3}\overset{2}{a}{}^m+\{(\overset{1}{\mathrm{L}}{}^{\alpha j3k}-\overset{0}{\mathrm{L}}{}^{\alpha m3r}(\overset{0}{\mathrm{L}}{}^{3q3r})^{-1}\overset{1}{\mathrm{L}}{}^{3q3k})$$

$$(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{a}{}^{m}\},\quad \overset{1}{a}+\overset{1}{\Gamma}{}^{j}_{m3}\overset{1}{a}{}^{m}]\,d\xi \tag{26}$$

where,

$$\begin{aligned}\overset{3}{G}{}^{\alpha j\beta l}=&\overset{3}{L}{}^{\alpha j\beta l}\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{3}{L}{}^{3m\beta l}-\overset{3}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{0}{L}{}^{3m\beta l}+\overset{0}{L}{}^{\alpha j3k}\\&(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{2}{L}{}^{3q\beta l}+\overset{2}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3n}\\&(\overset{0}{L}{}^{3p3n})^{-1}\overset{0}{L}{}^{3p\beta l}-\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q3p}\\&(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3r\beta l}-\overset{1}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3r3n}\\&(\overset{0}{L}{}^{3s3n})^{-1}\overset{0}{L}{}^{3s\beta l}+\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q3p}\\&(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3r3n}(\overset{0}{L}{}^{3s3n})^{-1}\overset{0}{L}{}^{3s\beta l}-\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3t}\\&(\overset{0}{L}{}^{3q3t})^{-1}\overset{2}{L}{}^{3q3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{0}{L}{}^{3r\beta l}-\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{2}{L}{}^{3m3t}\\&(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q3n}(\overset{0}{L}{}^{3p3n})^{-1}\overset{0}{L}{}^{3p\beta l}+\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{2}{L}{}^{3m3t}\\&(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q\beta l}+\overset{1}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{2}{L}{}^{3m3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{0}{L}{}^{3r\beta l}\\&+\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{3}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{0}{L}{}^{3q\beta l}-\overset{1}{L}{}^{\alpha j3k}\\&(\overset{0}{L}{}^{3m3k})^{-1}\overset{2}{L}{}^{3m\beta l}-\overset{2}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m\beta l}+\overset{1}{L}{}^{\alpha j3k}\\&(\overset{0}{L}{}^{3m3k})^{-1}\overset{1}{L}{}^{3m3p}(\overset{0}{L}{}^{3r3p})^{-1}\overset{1}{L}{}^{3r\beta l}\end{aligned}$$

and

$$\begin{aligned}\overset{2}{a}{}^{m}=\int_{-1/2}^{\xi}\{&(\overset{0}{G}{}^{\gamma m\beta l}\overset{2}{u}_{l,\beta}+\overset{1}{G}{}^{\gamma m\beta l}\overset{1}{u}_{l,\beta}+\overset{2}{G}{}^{\gamma m\beta l}\overset{0}{u}_{l,\beta})_{,\overset{0}{\gamma}}+\overset{1}{\Gamma}{}^{\gamma}_{nr}(\overset{0}{G}{}^{nm\beta l}\overset{1}{u}_{l,\beta}\\&+\overset{1}{G}{}^{m\beta l}\overset{0}{u}_{l,\beta})+\overset{1}{\Gamma}{}^{m}_{nr}(\overset{0}{G}{}^{\gamma n\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\gamma n\beta l}\overset{0}{u}_{l,\beta})+\overset{2}{\Gamma}{}^{\gamma}_{nr}\overset{0}{G}{}^{nm\beta l}\overset{0}{u}_{l,\beta}\\&+\overset{2}{\Gamma}{}^{m}_{ny}\overset{0}{G}{}^{\gamma n\beta l}\overset{0}{u}_{l,\beta}-(\overset{0}{L}{}^{\gamma m\beta l}(\overset{0}{L}{}^{3n3k})^{-1}\overset{1}{a}{}^{n})_{,\overset{0}{r}}-\overset{0}{\Gamma}{}^{m}_{n3}\overset{1}{a}{}^{n}\}\,d\xi\end{aligned}$$

and the solution is

$$\begin{aligned}\partial\overset{4}{u}_l/\partial\xi=&-(\overset{0}{L}{}^{3m3l})^{-1}[\overset{2}{a}{}^{m}-\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{a}{}^{q}+\{\overset{3}{L}{}^{3m\beta k}-\overset{1}{L}{}^{3m3t}\\&(\overset{0}{L}{}^{3q3t})^{-1}\overset{2}{L}{}^{3q\beta k}+\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q3p}(\overset{0}{L}{}^{3r3p})^{-1}(\overset{1}{L}{}^{3r\beta k}-\\&\overset{1}{L}{}^{3r3n}(\overset{0}{L}{}^{3s3n})^{-1}\overset{0}{L}{}^{3s\beta k})+\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{2}{L}{}^{3q3p}(\overset{0}{L}{}^{3r3p})^{-1}\\&\overset{0}{L}{}^{3r\beta k}-\overset{2}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}(\overset{1}{L}{}^{3q\beta k}-\overset{1}{L}{}^{3q3n}(\overset{0}{L}{}^{3p3n})^{-1}\overset{0}{L}{}^{3p\beta k})-\\&\overset{3}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{0}{L}{}^{3q\beta k}\}\overset{0}{u}_{k,\beta}+\{\overset{2}{L}{}^{3m\beta k}-\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\\&\overset{1}{L}{}^{3q\beta k}+\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{1}{L}{}^{3q3p}(\overset{0}{L}{}^{3r3t})^{-1}\overset{0}{L}{}^{3r\beta k}-\overset{2}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\\&\overset{0}{L}{}^{3q\beta k}\}\overset{1}{u}_{k,\beta}+\{\overset{1}{L}{}^{3m\beta k}-\overset{1}{L}{}^{3m3t}(\overset{0}{L}{}^{3q3t})^{-1}\overset{0}{L}{}^{3q\beta k}\}\overset{2}{u}_{k,\beta}+\\&\overset{0}{L}{}^{3m\beta k}\overset{3}{u}_{k,\beta}-\overset{0}{L}{}^{3m3k}(\overset{0}{u}_n\overset{3}{\Gamma}{}^{n}_{3k}+\overset{1}{u}_n\overset{2}{\Gamma}{}^{n}_{k3}+\overset{2}{u}_n\overset{1}{\Gamma}{}^{n}_{k3}+\overset{3}{u}_n\overset{0}{\Gamma}{}^{n}_{k3})]\end{aligned}$$

The bifurcation functional for the case where the hydrodynamic pressure is applied on the surfaces of the structure will have the additional pressure term, i.e., for the updated Lagrangian description, the bifurcation functional is

$$\begin{aligned}F(\lambda,\Delta\dot{u}_i)=&\int_V\{(L^{ijkl}+\sigma^{ik}g^{jl})\Delta\dot{u}_{j,i}\Delta\dot{u}_{l,k}\}dV\\&+\int_{\partial V}p(g^{ij}g^{kl}-g^{il}g^{jk})\Delta\dot{u}_{l,k}\Delta\dot{u}_j n_i dS\end{aligned}\tag{27}$$

where p is the hydrodynamic pressure on the surface ∂V of the structure and so it is a function of the thickness coordinate ζ as well as the midgeometry coordinate θ^α, i.e., $p=p(\theta^\alpha,\zeta)$. And the corresponding bifurcation equation and the boundary condition is

$$\begin{aligned}&\{(L^{ijkl}+\sigma^{ik}g^{jl})\Delta\dot{u}_{l,k}\}_{,i}=0\quad\text{in } V\\&(L^{ijkl}+\sigma^{ik}g^{jl})\Delta\dot{u}_{l,\beta}n_i=-p(g^{ij}g^{kl}-g^{il}g^{jk})\Delta u_{l,k}n_i\\&\qquad\text{on } \partial V\end{aligned}\tag{28}$$

For the notational convenience, we define

$$\begin{aligned}&L^{ijkl}\equiv L^{ijkl}+\sigma^{ik}g^{jl}=L^{klij}\\&N^{ijkl}\equiv p(g^{ij}g^{kl}-g^{ij}g^{jk})=N^{klij}\end{aligned}\tag{29}$$

Following the same analysis procedure, we will obtain the straightforward asymptotic solutions for the constant solid thickness, i.e.,

The lowest order solution is,

$$(\overset{0}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\overset{0}{\alpha}}=\overset{0}{F}_1{}^{3j\beta l}\overset{0}{u}_{l,\beta}\tag{30}$$

with

$$\begin{aligned}&\overset{0}{G}{}^{\alpha j\beta l}=\overset{0}{L}{}^{\alpha j\beta l}-\overset{0}{L}{}^{\alpha j3k}(\overset{0}{L}{}^{3m3k})^{-1}\overset{0}{L}{}^{3m\beta l}\\&\overset{0}{F}{}^{ij\beta l}=\overset{1}{N}{}^{ij\beta l}-\overset{1}{N}{}^{ij3k}(\overset{0}{L}{}^{3r3k})^{-1}\overset{0}{L}{}^{3r\beta l}\equiv\overset{0}{F}_1{}^{ij\beta l}\xi+\overset{0}{F}_2{}^{ij\beta l}\end{aligned}$$

and $\overset{0}{u}_1=\overset{0}{w}_1(\theta^\alpha)$

The first order solution is,

$$\begin{aligned}&\int_{-1/2}^{1/2}\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{1}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\overset{0}{\alpha}}+(\overset{1}{\Gamma}{}^{\alpha}_{ma}\overset{0}{G}{}^{mj\beta l}+\overset{1}{\Gamma}{}^{j}_{ma}\overset{0}{G}{}^{\alpha m\beta l}+\overset{0}{\Gamma}{}^{j}_{m3}\\&\overset{1}{U}{}^{3m\beta l})\overset{0}{u}_{l,\beta}\}d\xi-(\overset{0}{F}_1{}^{3j\beta l}\overset{1}{w}_{l,\beta}+\overset{0}{F}_2{}^{3j\beta l}\overset{1}{B}_{l,\beta}+\overset{1}{F}_2{}^{3j\beta l}\overset{0}{u}_{l,\beta})=0\end{aligned}\tag{31}$$

with

$$\begin{aligned}&\overset{1}{G}{}^{ijkl}=\overset{1}{L}{}^{ijkl}-\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}-\overset{1}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl}+\\&\qquad\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}(\overset{0}{L}{}^{3q3p})^{-1}\overset{0}{L}{}^{3qkl}\\&\overset{1}{U}{}^{ijkl}=\overset{0}{L}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{F}{}^{3nkl},\quad\overset{\underline{1}}{G}{}^{ijkl}=\overset{1}{G}{}^{ijkl}+\overset{1}{U}{}^{ijkl}\end{aligned}$$

and

$$\begin{aligned}\overset{1}{F}{}^{ijkl}=&\overset{2}{N}{}^{ijkl}-\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3nkl}+\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{1}{L}{}^{3n3p}\\&(\overset{0}{L}{}^{3q3p})^{-1}\overset{0}{L}{}^{3qkl}-\overset{2}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{L}{}^{3nkl}\\\overset{1}{V}{}^{ijkl}=&-\overset{1}{N}{}^{ij3m}(\overset{0}{L}{}^{3n3m})^{-1}\overset{0}{F}{}^{3nkl},\quad\overset{\underline{1}}{F}{}^{ijkl}=\overset{1}{F}{}^{ijkl}+\overset{1}{V}{}^{ijkl}\end{aligned}$$

And also,

$$\overset{1}{u}_1(\theta^\alpha,\xi)=-(\overset{0}{L}{}^{3j3l})^{-1}(\overset{0}{L}{}^{3j\beta k}\overset{0}{u}_{k,\beta}-\overset{0}{L}{}^{3j3k}\overset{0}{u}_n\overset{0}{\Gamma}_{k3}{}^{n})\xi+\overset{1}{w}_i(\theta^\alpha)\tag{32}$$

Now, every mode is not unique in the function space, i.e., we have the infinite modes, but the practical solution is unique. And hence, we should have the proper mode orthogonality condition for the uniqueness of the solution, as

$$\int_V\overset{0}{v}_j\overset{i}{v}_j dV=0\quad(i\geq 1)\tag{33}$$

in terms of the physical components of the buckling mode. The physical component is defined as

$$v_j\equiv\Delta\dot{u}_j\sqrt{g^{ll}}\,(\text{no sum})=\overset{0}{v}_j+\overset{1}{v}_j\varepsilon+\overset{2}{v}_j\varepsilon^2+\overset{3}{v}_j\varepsilon^3+\cdots\tag{34}$$

We may note that the orthogonality condition (33) is not unique. That is, if the lowest order mode is zero, we cannot use (33). In such a case, the following condition should be used

$$\int_V\overset{1}{v}_j\overset{i}{v}_j dV=0\quad(i\geq 2)\ \text{ with }\overset{1}{v}_j\neq 0\tag{35}$$

The meaning of the above mode orthogonality condition is that every mode should be orthogonal to the basic mode (the lowest nontrivial mode) in the admissible function space for the uniqueness of the solution.

4. CONCLUSIONS

The purpose of this work in Part I is to propose a consistent unified method for the analysis of buckling instabilities in solids of the arbitrary thickness. Instead of the classical nonlinear buckling theories, in the present approach the full three dimensional linearized bifurcation equations for the solid are asymptotically expanded with respect to its thickness.

The natural selection for the small parameter is the reference solid thickness which is assumed to be constant versus some characteristic length(L) of the reference geometry. In the limiting process considered, the reference middle geometry remains fixed as the initial reference thickness tends to zero, thus making the limiting process dependent only on one small parameter.

This unified methodology results in the solution of a sequence of two dimensional boundary value problems whose domain is the reference middle geometry of the solid. The theoretical advantages in comparison with the classical approach of the proposed method are essentially three : the consistency in obtaining the critical loads and modes independently of the nonlinear solid theory employed, the possiblility of obtaining higher order terms analytically if necessary for the thickness dependence of those quantities, and the mathematical foundation for the development of general purpose numerical codes (finite element codes) for the solid buckling analysis.

In the forthcoming Part Ⅱ, the application of this general development to a simple structure will be treated.

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